ON THE PROBLEM OF MANY-BODY LOCALIZATION

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Abstract

We review recent progress in the study of transport properties of inter-
acting electrons subject to a disordered potential which is strong enough
to localize *all* single-particle states. This review may also serve as a guide
to the recent paper by the authors [Annals of Physics (2006), in press].
Here we skip most of the technical details and make an attempt to discuss
the physical grounds of the final-temperature metal-insulator transition
described in the above-mentioned paper.

1 Introduction

Transport properties of conducting materials at low temperature T are deter-
mined by an interplay between the interaction of the itinerant electrons with
each other and the quenched disorder which creates a random potential act-
ing on these electrons. In the absence of the electron-electron interaction the
most dramatic phenomenon is Anderson localization (Anderson, 1958) – the dc
electrical conductivity σ can be qualitatively different depending on whether one-
particle wave functions of the electrons are localized or not. In the latter case
$\sigma(T)$ has a finite zero-temperature limit, while in the former case $\sigma(T)$ vanishes
when $T \to 0$. Therefore, Anderson localization of electronic states leads to the
Metal to Insulator Transition at zero temperature.

When discussing zero temperature conductivity $\sigma(0)$, we need to consider
only electronic states close to the Fermi level. The conductivity becomes finite
at any finite temperature provided that extended states exist somewhere near
the Fermi level. It is commonly accepted now that localized and extended states
in a random potential can not be mixed in the one-electron spectrum and thus
this spectrum in a general case is a combination of bands of extended states
and bands of localized states. A border between a localized and an extended
band is called mobility edge. If the Fermi level is located inside a localized band
and inelastic scattering of the electrons is completely absent, the conductivity
should follow Arrhenius law $\sigma(T) \propto \exp(-E_c/T)$, where E_c is the distance from
the Fermi level to the closest mobility edge. Another common belief following
from the scaling theory of Anderson localization (Thouless, 1977; Abrahams *et
al.*, 1979), is that in low dimensionality d, namely at $d = 1, 2$ all states are
localized in an arbitrarily small disorder, while for free electrons (no periodic

potential) $E_c > 0$ is finite at $d = 3$.[1] It means that without inelastic processes $\sigma_{d=1,2}(T) = 0$, while for $\sigma_{d=3}(T)$ one should expect the Arrhenius law. Note that for electrons in a crystal within a given conduction band the latter conclusion is not always correct – strong enough disorder can localize the whole band.

As soon as inelastic processes are included, the situation becomes more complicated. In particular, electron-phonon interaction leads to the mechanism of conductivity known as hopping conductivity (Fritzche, 1955; Mott, 1968a; Shklovskii and Efros, 1984) – with an assistance of phonons, electrons hop between the localized states without being activated above the mobility edge. As a result, $\sigma(T)$ turns out to be finite (although small) at arbitrarily low T even when all one-electron states are localized.

Can interaction between electrons play the same role and cause the hopping conductivity? This question was discussed in literature for a long time (Fleishman and Anderson, 1980; Shahbazyan and Raikh, 1996; Kozub, Baranovskii, and Shlimak, 2000; Nattermann, Giamarchi, and Le Doussal, 2003; Gornyi, Mirlin, and Polyakov, 2004) and no definite conclusion was achieved. The problem is that although the electric noise exists inside the material with a finite ac conductivity[2] the "photons" in contrast with phonons become localized together with electrons.

In a recent work (Basko, Aleiner, and Altshuler, 2006) we have demonstrated that electron-electron interaction alone cannot cause finite conductivity even when temperature is finite, but small enough. In the absence of phonons and extended one-electron states conductivity of a system of interacting electrons vanishes exactly below some critical temperature T_c. At the same time, at high temperatures $T > T_c$ the conductivity $\sigma(T)$ is finite. It means that at $T = T_c$ the system of interacting electrons subject to a random potential undergoes a genuine phase transition that manifests itself by the emerging of a finite conductivity!

This transition can be thought of as many-body localization – it applies to many-body eigenstates of the whole system. This localization occurs not in the real space, but rather in the Fock space. This fact does not affect the validity of the concept of mobility edge. In fact, the existence of the "metallic" state at $T > T_c$ implies that the many-body states with energies $\mathcal{E}$ above $\mathcal{E}_c$ are extended. One can estimate the difference between $\mathcal{E}_c$ and the energy of the many-body ground state $\mathcal{E}_0$ as $\mathcal{E}_c - \mathcal{E}_0 \sim T\mathcal{N}(T)$, where $\mathcal{N}(T)$ is the total number of one-particle states in the energy strip of the width T. Note that the existence of the extended many-body states above the mobility edge does not contradict the fact that below T_c there is no conductivity – in contrast with the case of one-particle localization there is no Arrhenius regime since $\mathcal{E}_c - \mathcal{E}_0$ turns out to be proportional to the volume of the system, *i.e.*, is macroscopically large (see Sec. 4 below for more details).

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[1] For $d = 1$ this statement was proved rigorously both for one-channel (Gertsenshtein and Vasil'ev, 1959; Berezinskii, 1973) and multi-channel (Efetov and Larkin, 1983; Dorokhov, 1983) disordered wires.

[2] In this paper we mostly focus on dc conductivity. As to ac conductivity, it never vanishes, because at any frequency density of resonant pairs of states is finite.

In order to avoid possible misunderstanding we would like to emphasize that we focus only on the inelastic collisions between the electrons, *i.e.*, on creation or annihilation of *real* electron-hole pairs. There are other effects of electron-electron interactions which can be understood as renormalization of the one-particle random potential by the interaction. Being temperature-dependent, this renormalization leads to a number of interesting effects, such as the interaction corrections to the density of states and conductivity in disordered metals (Altshuler and Aronov, 1985). On the insulating side of the one-particle localization transition similar effects cause the well-known Coulomb gap (Shklovskii and Efros, 1984) which reduces hopping conductivity. On the other hand, this is just a correction to the time-independent random potential. As such, it can maybe shift the position of the many-body Metal to Insulator transition, *i.e.*, renormalize T_c, but is unable to destabilize the insulating or metallic phases. From now on we will simply neglect all elastic (Hartree-Fock) effects and concentrate on the *real* inelastic electron-electron collisions.

Localization of the many-body states in the Fock space has been discussed by Altshuler *et al.* (1997) for the case of zero-dimensional systems with finite, although large, number of electrons. In this paper the authors proposed an approximate mapping of the Hamiltonian of a metallic grain with large Thouless conductance g and moderate interaction between the electrons to the one-particle Hamiltonian on a lattice with the topology of the Cayley tree and an on-site disorder. The latter problem has an exact solution (Abou-Chacra, Anderson, and Thouless, 1973; Efetov, 1987) that exhibits the localization transition. In terms of interaction electrons this transition means that one-particle excitation states below certain energy are quite close to some exact many-body excitations. As to the one-particle excitations with energies higher than the critical one, its wave function can be viewed as a linear combination of a large number of the many-body eigenstates.

For an infinite system ($d > 0$) the situation is more complex, and Cayley tree approximation is hard to justify. Nevertheless, a consistent analysis of a model with weak and short range interaction to all orders of perturbation theory enabled us to analyze the many-body localization transition and to demonstrate that both the metallic state at high temperatures and the insulating state at low temperatures are stable and survive all higher loop corrections to the locator expansion. Therefore, the existence of the transition is proved on the physical level of rigor.

It should be noted that such an insulating state that is characterized by exactly zero conductivity is quite different from all other known types of insulators. For example, Mott insulator is believed to have finite, though exponentially small conductivity at finite temperatures.

The present text represents a shortened version of the paper by Basko, Aleiner, and Altshuler (2006), hereafter referred to as BAA paper. We omit most of the technical details (for which the reader will be referred to specific sections of the BAA paper), and stress the key ideas.

The remainder of the paper is organized as follows. In Sec. 2 we briefly review some well-known facts about electric conduction in Anderson insulators and pose the problem. Sec. 3 represents a sketch of the solution whose details are given in BAA paper. We discuss the model for interacting localized electrons in Sec. 3.1, and the corresponding Fock space picture in Sec. 3.2. In Sec. 3.3 we show the formal way to characterize metallic and insulating phases. In Sec. 3.4 we introduce the main approximation used in the calculation (self-consistent Born approximation), and discuss its validity. The existence of the metallic state at high temperatures and its properties are discussed in Sec. 3.5. Sec. 3.6 is dedicated to the proof of existence of the insulating phase at low temperatures; the value of the transition temperature is obtained as the limit of stability of the insulating phase. In Sec. 4 we discuss the macroscopic implications of the problem, introducing the concepts of many-body localization and many-body mobility edge. Finally, in Sec. 5 we summarize the results and present an outlook of the future developments.

2   Background and formulation of the problem

2.1   *Non-interacting electrons in disorder potential*

Let us briefly review the basic concepts developed for the problem of one-electron wave functions in a disordered potential in d dimensions. Depending on the strength of the disorder potential, a wave function $\phi_\alpha(\vec{r})$ of an eigenstate α with the energy ξ_α can be either *localized* or *extended*:

$$|\phi_\alpha(\vec{r})|^2 \propto \begin{cases} \frac{1}{\zeta_{loc}^d} \exp\left(-\frac{|\vec{r}-\vec{\rho}_\alpha|}{\zeta_{loc}}\right), & \text{localized;} \\ \frac{1}{\Omega}, & \text{extended.} \end{cases} \tag{1}$$

Here ζ_{loc} is the localization length which depends on the eigenenergy ξ_α, and Ω is the volume of the system. Each localized state is characterized by a point in space, $\vec{\rho}_\alpha$, where $|\phi_\alpha(\vec{r})|^2$ reaches its maximum, and an exponentially falling envelope. Extended states spread more or less uniformly over the whole volume of the system. Localized and extended states cannot coexist at the same energy, and the spectrum splits into bands of localized and extended states. The energies separating such bands are known as *mobility edges*. For free electrons in $d \geq 3$ disorder potential leads to only one mobility edge $\mathcal{E}_1$, so that

$$\begin{aligned} \xi_\alpha < \mathcal{E}_1 : & \quad \text{localized;} \\ \xi_\alpha > \mathcal{E}_1 : & \quad \text{extended.} \end{aligned} \tag{2}$$

If a finite mobility edge (2) exists and the Fermi level ϵ_F lies in the band of localized states, the conductivity is determined by the exponentially small occupation number of the delocalized states

$$\sigma(T) \propto e^{-(\mathcal{E}_1 - \epsilon_F)/T}. \tag{3}$$

In this paper we are interested in transport properties of the systems where *all* single-particle states are localized, and thus without many-body effects $\sigma = 0$

at any temperature. It is well established now that the mobility edge usually does not exist for one- and two-dimensional systems, and all single-particle states are indeed localized for an arbitrarily weak disorder. Such a situation can arise for a large d as well, if the bandwidth is finite and disorder is sufficiently strong.

2.2 *Role of inelastic processes and phonon-assisted hopping*

As long as all single-particle states are localized, transport occurs only because of inelastic processes, which transfer electrons between different localized eigenstates. At this stage we introduce the main energy scale of the problem: the typical energy spacing between states whose spatial separation does not exceed ζ_{loc}, so that there is overlap between their wave functions:

$$\delta_\zeta = \frac{1}{\nu \zeta_{loc}^d} \,, \tag{4}$$

where ν is the one-particle density of states per unit volume.

The conductivity is, roughly speaking, proportional to the rate of the transitions between different localized states, which can be called inelastic relaxation rate Γ. Obviously, at $T = 0$ inelastic processes disappear, so regardless of the mechanism of inelastic relaxation the conductivity must vanish:

$$\lim_{T \to 0} \sigma(T) = 0 \,. \tag{5}$$

The question is how $\sigma(T)$ approaches zero for each particular mechanism.

When phonons are the main source of inelastic scattering, the answer is given by Mott's variable range hopping formula (Mott, 1968a)[3]

$$\sigma(T) = \sigma_0 \left(\frac{T}{\delta_\zeta}\right)^\alpha \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{1/(d+1)}\right], \tag{6}$$

where σ_0 and α are constants. According to Eq. (6), $\sigma(T)$ remains finite as long as $T \neq 0$. The reason for $\sigma(T) \neq 0$ is that for any pair of localized states one can always find a phonon whose frequency exactly corresponds to their energy mismatch, at low temperature one should just wait long enough.

The same type of $\sigma(T)$-dependence would result from the coupling of electrons with *any* delocalized thermal bath whose energy spectrum is continuous down to zero energy. The specific nature of the bath at most affects the power-law prefactor. On the contrary, the stretched exponential factor is universal; it originates from the counting of electronic states available for the transition, and does not depend on the specific scattering mechanism.

2.3 *Inelastic relaxation due to electron-electron interaction*

Now let us assume that there is no external bath coupled to electrons, but some electron-electron interaction is present. What will be the dependence of $\sigma(T)$?

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[3]Here we do not consider effects of the Coulomb interaction which are known to modify the power of temperature in the exponent (Shklovskii and Efros, 1984).

In line with the discussion of the previous subsection, the question should be posed as follows: do electron-hole pairs themselves provide a suitable bath in a localized system, thus validating Eq. (6)?

One possible answer is "yes". Indeed, recall Mott formula for the low-temperature dissipative ac conductivity $\sigma(\omega)$ in a localized system (Mott, 1968b):

$$\sigma(\omega) = \sigma_1 \frac{\omega^2}{\delta_\zeta^2} \ln^{d+1} \frac{\delta_\zeta}{|\omega|} \,. \tag{7}$$

According to fluctuation-dissipation theorem, at finite temperature electromagnetic fluctuations of a finite spectral density should be present, and they might serve as a bath. The problem with this argument is that Eq. (7) is the *spatial average* of the conductivity over the whole (infinite) volume. For each given realization of disorder, excitations determining $\sigma(\omega)$ from Eq. (7) are localized, and although the total volume is infinite, the spectrum of electron-hole pairs is effectively discrete.

The crucial point is that as long as electron-electron interaction is local in space (here we do not consider long-range interactions), it effectively couples electronic states only within the same localization volume, where the spectrum of electronic states is effectively discrete. The following sections are dedicated to a systematic discussion of this problem which was first pointed out by Fleishman and Anderson (1980). The conclusion can be stated as follows: electron-hole excitations can cause finite conductivity only if the temperature of the system exceeds some critical value. At lower temperatures $\sigma(T)$ vanishes exactly.

3   Finite-temperature metal-insulator transition

3.1   *Matrix elements of electron-electron interaction between localized states: essential features of the model*

For simplicity we consider a system of spinless electrons and assume that electron-electron interaction is weak and short-range:[4]

$$V(\vec{r}_1 - \vec{r}_2) = \frac{\lambda}{\nu} \, \delta(\vec{r}_1 - \vec{r}_2) \,, \tag{8}$$

where $\lambda \ll 1$ is the dimensionless interaction constant, ν is the one-particle density of states per unit volume.

In the basis of localized single-particle eigenstates the hamiltonian corresponding to the pair interaction potential (8) takes the form

$$\hat{H} = \sum_\alpha \xi_\alpha \hat{c}_\alpha^\dagger \hat{c}_\alpha + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} \, \hat{c}_\alpha^\dagger \hat{c}_\beta^\dagger \hat{c}_\gamma \hat{c}_\delta \tag{9}$$

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[4]Interaction proportional to $\delta(\vec{r}_1 - \vec{r}_2)$ in the strict sense is equivalent to no interaction for spinless electrons, considered here, due to the Pauli principle. Here, by writing $\delta(\vec{r}_1 - \vec{r}_2)$ we only mean that the range is much smaller than the electron mean free path, so it is not a true δ-function.

Consider the structure of the matrix elements $V_{\alpha\beta\gamma\delta}$. Since $V(\vec{r})$ is short-range, they decrease exponentially when the spatial separation between the states increases, the characteristic scale being the localization length ζ_{loc}. In addition to this spatial suppression, the matrix elements decrease rapidly when the energy difference, say $\xi_\alpha - \xi_\gamma$, increases exceeding the level spacing δ_ζ. This occurs because the localized wave functions oscillate randomly, and the bigger the energy difference, the weaker are these random oscillations correlated (Altshuler and Aronov, 1985). Provided that the restrictions

$$|\vec{r}_\alpha - \vec{r}_\beta| \lesssim \zeta_{loc}, \quad |\vec{r}_\alpha - \vec{r}_\gamma| \lesssim \zeta_{loc}, \quad |\vec{r}_\beta - \vec{r}_\gamma| \lesssim \zeta_{loc}, \quad \text{etc.,} \tag{10}$$

$$|\xi_\alpha - \xi_\delta|, |\xi_\beta - \xi_\gamma| \lesssim \delta_\zeta \quad \text{or} \quad |\xi_\alpha - \xi_\gamma|, |\xi_\beta - \xi_\delta| \lesssim \delta_\zeta, \tag{11}$$

are fulfilled, we have $|V_{\alpha\beta\gamma\delta}| \sim \lambda\delta_\zeta$.

There are several ways to model these essential properties of $V_{\alpha\beta\gamma\delta}$. In BAA paper a specific model was adopted; essentially, the space and energy dependences of the matrix elements were replaced by simple rectangular cutoffs (see Sec. 3 of BAA paper for details).

3.2 *Many-electron transitions and Fock space*

We wish to note that all the discussion of this subsection is not conceptually new. In fact, it is just a generalization of the arguments of Altshuler *et al.* (1997) to an infinite system.

Conventionally, an elementary inelastic process is a decay of one single-particle excitation (an electron occupying a state α) into three single-particle excitations – a hole in the state β and two electrons in the states γ and δ. Such a decay can be described differently: one can say that the hamiltonian couples the single-particle excitation with the three-particle excitation by the matrix element $V_{\alpha\beta\gamma\delta}$. Further action of the interaction hamiltonian produces five-particle excitations, seven-particle excitations, *etc.*:

$$\xi_\alpha \;\to\; \xi_\gamma + \xi_\delta - \xi_\beta \;\to\; \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \;\to\ldots. \tag{12}$$

If on each stage the coupling is strong enough (*i. e.*, the matrix element is of the same order or larger than the corresponding energy mismatch), the single-particle excitation indeed decays irreversibly into all possible many-body states. In other words, exact many-body eigenstates become delocalized in the Fock space. If, oppositely, three-particle states contribute only a weak perturbative admixture to the one-particle state, the contribution from five-particle states is even weaker, *etc.*, the initial electron will never decay completely. One can say that it is localized in the Fock space.

Another way to visualize the inelastic relaxation is to look at the energy structure of the quasiparticle spectral function:

$$A_\alpha(\epsilon) = \sum_k \left| \langle \Psi_k | \hat{c}_\alpha^\dagger | \Psi_0 \rangle \right|^2 \delta(\epsilon + E_0 - E_k). \tag{13}$$
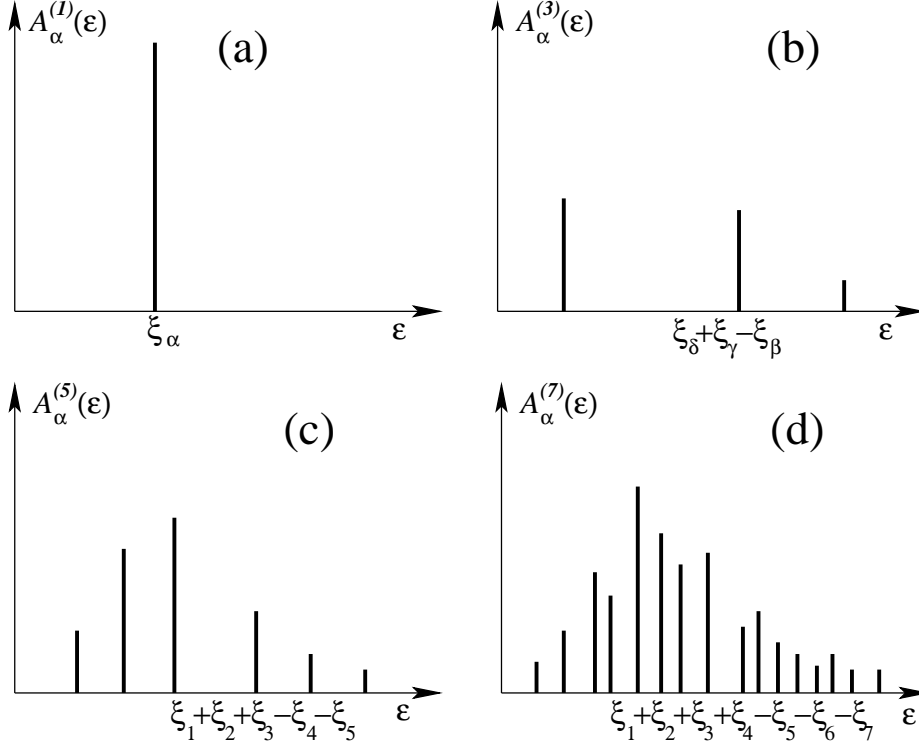
FIG. 1. A schematic view of the contributions to the perturbative expansion (14) of the spectral function $A_\alpha(\epsilon)$ from one-particle, three-particle, five-particle, and seven-particle excitations.

Here Ψ_0 and Ψ_k are many-body eigenstates (Ψ_0 is not necessarily a ground state), E_0 and E_k are the corresponding energies. Basically, $A_\alpha(\epsilon)$ shows how the single-particle excitation on top of a given eigenstate is spread over other many-body eigenstates of the system.

One can represent the spectral function in the form of expansion in powers of the interaction constant λ (Fig. 1):

$$A_\alpha(\epsilon) = \sum_{n=0}^{\infty} \lambda^{2n} A_\alpha^{(2n+1)}(\epsilon).\tag{14}$$

$A_\alpha^{(1)}(\epsilon)$ corresponds to the bare quasiparticle peak, $\delta(\epsilon - \xi_\alpha)$. Linear in λ term represents a random Hartree-Fock shift of the energy ξ_α which is already assumed to be random. This linear term is of no interest to us, so it is not included in the expansion (14). The λ^2 term corresponds to the contribution of three-particle excitations. The number of these excitations is effectively finite due to the restrictions (10), (11), so $A^{(3)}(\epsilon)$ is a collection of δ-peaks at three-particle energies

$\xi_\gamma + \xi_\delta - \xi_\beta$. Again, λ^3 terms correspond only to peak shifts and are omitted. The five-particle contribution $A^{(5)}(\epsilon)$ is again a collection of δ-peaks, however, spaced more closely than three-particle peaks. This represents the general rule: the more particles are involved, the higher is the density of the peaks. At the same time, contributions of many-particle processes are suppressed due to the smallness of λ. What is crucial to us, is the result of this competition at $n \to \infty$: will the empty spaces between peaks be filled making $A_\alpha(\epsilon)$ a continuous function, or the process will be suppressed by the powers of λ and $A_\alpha(\epsilon)$ will remain a collection of δ-peaks?

Thus, the problem of inelastic relaxation of a quasiparticle due to electron-electron interaction is related to the problem of localization or delocalization of excitations in the many-electron Fock space. The simplest model describing localization-delocalization physics is Anderson model. Anderson (1958) considered a tight-binding model on a d-dimensional lattice. The coupling between sites is nearest-neighbor only with a fixed matrix element V. The on-site energies are assumed to be random and uncorrelated, with some typical value denoted by W.

It is thus tempting to identify the many-electron hamiltonian (9) with the Anderson hamiltonian on a certain lattice, whose sites correspond to many-particle excitations (4). The approximate rules of correspondence then should be the following:

- $V \to \lambda\delta_\zeta$ – typical value of the coupling matrix element;
- $W \to \delta_\zeta$ – typical energy mismatch in each consecutive virtual transition, $|\xi_\alpha + \xi_\beta - \xi_\gamma - \xi_\delta| \sim \delta_\zeta$;
- finally, the coordination number $2d \to T/\delta_\zeta$. This represents the number of three-particle excitations to which a given single-particle excitation is coupled, with the energy mismatch not exceeding δ_ζ. The value T/δ_ζ is obtained as the product of the number of electrons β within the same localization volume, available for the collision with the probe electron α ($\sim T/\delta_\zeta$), and the number of ways to distribute energy allowed by the restriction (11), which is ~ 1.

According to Anderson (1958), localization-delocalization transition occurs at

$$\frac{Vd}{W}\ln\frac{W}{V} \sim \frac{\lambda T}{\delta_\zeta}\ln\frac{1}{\lambda} \sim 1 \,. \tag{15}$$

For the interacting localized electrons this would imply that below some temperature T_c the inelastic relaxation is frozen, so that the conductivity vanishes exactly; above T_c some inelastic relaxation is taking place, and the conductivity is finite. This corresponds to a finite-temperature metal-insulator transition.

This analogy, however, should be used with caution. Anderson's result, in fact, can be sensitive to the structure of the lattice; possible sources of problems are listed in Sec. 2.2 of BAA paper. One can develop a systematic approach to the problem, based on the diagrammatic technique for interacting electrons (Basko, Aleiner, and Altshuler, 2006). Below we headline the basic ideas of this
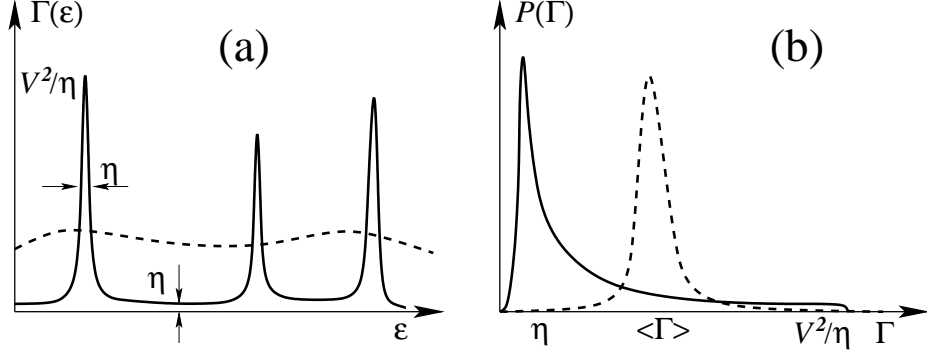
FIG. 2.   (a) Schematic energy dependence of the quasiparticle decay rate $\Gamma(\epsilon)$ in the metallic (dashed line) and insulating (solid line) phases for a given realization of single-particle levels $\{\xi_\alpha\}$. (b) The corresponding distribution functions $P(\Gamma)$ for $\Gamma(\epsilon)$ at a given energy ϵ in the metallic (dashed line) and insulating (solid line) phases.

approach to justify the assumption that Eq. (15) indeed correctly determines the point of the metal-insulator transition.

3.3   *Statistics of the transition rates*

We have to admit, that all the discussion of this subsection is not conceptually new either. It rather represents a generalization of the arguments of Anderson (1958) to a many-body problem.

As discussed in previous sections, the focus of the problem is the inelastic quasiparticle relaxation, which is represented by the imaginary part of the single-particle self-energy:

$$\Gamma_\alpha(\epsilon) = \operatorname{Im}\Sigma_\alpha^A(\epsilon)\,. \tag{16}$$

We stress that $\Gamma_\alpha(\epsilon)$ is a random quantity, as it depends on the positions of all single-particle levels $\{\xi_\beta\}$ and all occupation numbers $\{n_\beta\}$. One has no other choice but to perform statistical analysis of this random quantity.

How to distinguish between metallic and insulating phases within a statistical framework? It is clear that positions of the peaks in $\Gamma(\epsilon)$ for the insulating regime wander randomly with the variation of random energies ξ_α from Eq. (9). Due to this variation, the ensemble average of the decay rate $\langle\Gamma(\epsilon)\rangle$ is the same in both phases and can not be used for the distinction. In fact, one has to investigate the whole distribution function $P(\Gamma)$.

To understand the difference in the behavior of $P(\Gamma)$ in the two phases, it is instructive to start with the behavior of $\Gamma_\alpha(\epsilon)$ for a given realization of disorder, Fig. 2a. Deep in the metallic phase $\Gamma_\alpha(\epsilon)$ is a smooth function of energy, so its distribution $P(\Gamma)$ at a given energy is a narrow gaussian. In the insulating phase $\Gamma_\alpha(\epsilon)$ is given by a sequence of infinitely narrow δ-peaks, so an arbitrarily chosen
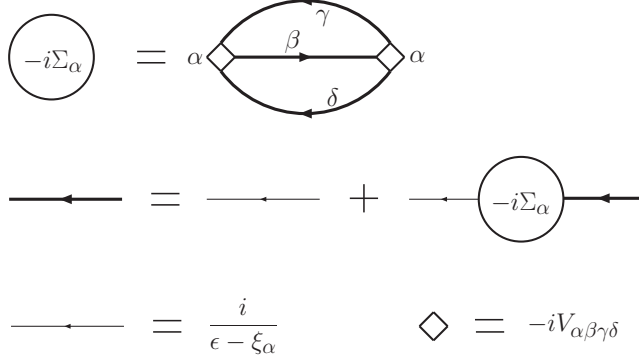
FIG. 3. Diagrammatic representation of self-consistent Born approximation.

value of ϵ falls between the peaks with the probability 1, giving $\Gamma = 0$. If ϵ hits a δ-peak, then $\Gamma = \infty$, which happens, however, with zero probability.

One can deal with this uncertainty by introducing an infinitesimal imaginary energy shift (damping) η, and consider $\Gamma_\alpha(\epsilon + i\eta)$. Physically, it may be viewed as an infinitesimally weak coupling to a dissipative bath (e. g., phonons). Clearly, in the metallic phase it has no effect, while in the insulating phase this damping broadens δ-peaks into lorentzians of the width η, which leads to appearance of the tails. Now, even if the energy ϵ falls between the peaks, Γ will have a finite value proportional to η. As a result, the distribution function $P(\Gamma)$ will have the form sketched in Fig. 2b. Now, having calculated the distribution function for a small but finite η, one can distinguish between the metallic and the insulating phases according to

$$\lim_{\eta \to 0} \lim_{\Omega \to \infty} P(\Gamma > 0) \begin{cases} > 0, & \text{metal} \\ = 0, & \text{insulator.} \end{cases} \tag{17}$$

Here Ω is the total volume of the system. We emphasize that the order of limits in Eq. (17) cannot be interchanged, since in a finite closed system the spectrum always consists of discrete δ-peaks.

3.4 *Self-consistent Born approximation*

Our main object of interest is the probability distribution function of the decay rate $P(\Gamma)$. Its calculation is performed in two stages: (i) we find $\Gamma_\alpha(\epsilon)$ for a given realization of disorder, (ii) we calculate its statistics. This subsection is dedicated to the first task.

We intend to describe both metallic and insulating regimes. In the latter regime relaxation dynamics is absent, the system never reaches thermal equilibrium, and temperature itself is ill-defined. Therefore, the only appropriate formal framework is the non-equilibrium formalism of Keldysh (1964) (see Sec. 4.1 of BAA paper for details).

Our approach for calculation of $\Gamma_\alpha(\epsilon)$ is the self-consistent Born approximation (SCBA), shown diagrammatically in Fig. 3, and given by

$$\Gamma_\alpha(\epsilon) = \eta + \pi \sum_{\beta,\gamma,\delta} |V_{\alpha\beta\gamma\delta}|^2 \int d\epsilon' \, d\omega \, A_\beta(\epsilon') \, A_\gamma(\epsilon' + \omega) \, A_\delta(\epsilon - \omega) \times$$
$$\times \left[n_\beta(1 - n_\gamma)(1 - n_\delta) + (1 - n_\beta)n_\gamma n_\delta \right], \tag{18}$$

$$A_\alpha(\epsilon) = \frac{1}{\pi} \frac{\Gamma_\alpha(\epsilon)}{(\epsilon - \xi_\alpha)^2 + \Gamma_\alpha^2(\epsilon)} \,. \tag{19}$$

Here $n_\alpha = 0, 1$ is the fermion occupation number of the single-particle state α. It is crucial to realize that one cannot replace n_α with its average equilibrium value, corresponding to Fermi-Dirac distribution. The basis in the Fock space is formed by Slater determinants corresponding to $n_\alpha = 0, 1$ only, and Γ represents the rate of a transition between two such basis states with different sets of $\{n_\alpha\}$. Temperature enters through total energy of the state, which is determined by all $\{n_\alpha\}$ and is proportional to the volume of the system (this issue will be discussed in more detail in Sec. 4 of the present paper).

Iterations of SCBA generate self-energy diagrams which have one common property: they describe decay processes where the number of particles in the final state is maximized for each given order in λ, thus maximizing the phase space available for the decay. Vice versa, each diagram satisfying this condition is taken into account by SCBA. Let us briefly discuss contributions which are neglected by Eqs. (18), (19); for a detailed discussion see Sec. 7 of BAA paper.

- Eqs. (18), (19) completely ignore the real part of the self-energy, $\text{Re}\,\Sigma$. In most of the terms effect of $\text{Re}\,\Sigma$ reduces to random uncorrelated corrections to already random energies, and appear to be completely negligible. In a more accurate approximation, however, some statistical correlations are present. They renormalize the numerical prefactor in the expression for the critical temperature, see Sec. 7.3 of BAA paper.

- Generally, quantum mechanical probability of a transition is given by a square of the total amplitude, the latter being a sum of partial amplitudes. In fact, Eqs. (18), (19) correspond to replacing the square of the sum by the sum of the squares. Approximation which neglects the interference terms can be justified in the same way as in the Anderson model of a high dimensionality $d_{eff} \sim T/\delta_\zeta \sim 1/\lambda \gg 1$, see Sec. 7.2 of BAA paper.

- Finally, diagrams generated by SCBA correspond to taking all n-particle vertex function in the leading approximation in λ. One can show that for $\lambda \ll 1$ vertex corrections are small, see Sec. 7.1 of BAA paper. For $\lambda \sim 1$ one would have to introduce the full n-particle vertex analogously to how it is done in conventional Fermi liquid theory for $n = 2$ (Landau, 1958; Eliashberg, 1962). In this case the result for the transition temperature will be determined not by the bare interaction constant, but by the statistics of the full vertex functions. Existence and regularity of these vertex functions

are basically equivalent to the assumption that in the absence of disorder the interacting system is a Fermi liquid.

3.5 *Metallic phase*

Starting from the self-consistent equations (18), (19), one can straightforwardly verify (see Sec. 5.1 of BAA paper) that $P(\Gamma)$ can be well approximated by gaussian distribution with the average and dispersion

$$\langle \Gamma \rangle \sim \lambda^2 T, \quad \langle \Gamma^2 \rangle - \langle \Gamma \rangle^2 \sim \lambda^2 \delta_\zeta^2, \tag{20}$$

provided that

$$\sqrt{\langle \Gamma^2 \rangle - \langle \Gamma \rangle^2} \ll \langle \Gamma \rangle \quad \Leftrightarrow \quad T \gg T^{(in)} \sim \frac{\delta_\zeta}{\lambda}. \tag{21}$$

According to the arguments of Sec. 3.3, this is characteristic of the metallic phase. One should not think, however, that this automatically means that the system has the same transport properties as conventional metals (which would mean that the conductivity is given by the Drude formula). The system conducts in the Drude regime only when the inelastic processes completely destroy the localization, the latter manifesting itself as the weak localization correction to conductivity (Altshuler and Aronov, 1985). This occurs when the discrete levels are completely smeared:

$$\langle \Gamma \rangle \gg \delta_\zeta \quad \Leftrightarrow \quad T \gg T^{(el)} \sim \frac{\delta_\zeta}{\lambda^2}. \tag{22}$$

It turns out that for $\lambda \ll 1$ there is a parametric range of temperatures:

$$\frac{\delta_\zeta}{\lambda} \ll T \ll \frac{\delta_\zeta}{\lambda^2}. \tag{23}$$

In this interval electron-electron interaction is already sufficient to cause inelastic relaxation, however, the localized nature of the single-particle wave functions remains important. Conduction in this regime can be viewed as an analog of hopping conduction discussed in Sec. 2.2, in the sense that electrons themselves indeed provide a good bath. However, as $T \gg \delta_\zeta$, there is no exponential factor in the temperature dependence $\sigma(T)$. This dependence can be obtained from the kinetic equation describing electron transitions between localized states, and is given by a power law, $\sigma(T) \propto T^\alpha$, where α can be model-dependent. Also, in this temperature range Wiedemann-Frantz law can be violated. For further details the reader is referred to Sec. 5.2 of BAA paper. We only note that this regime is somewhat analogous to the phonon-assisted conduction discussed by Gogolin, Mel'nikov, and Rashba (1975).

3.6 *Insulating phase*

Self-consistent equations (18), (19) represent a system of nonlinear integral equations whose coefficients are random due to randomness of level energies $\{\xi_\alpha\}$ and

occupation numbers $\{n_\alpha\}$. Apparently, for $\eta = 0$ these equations have a solution $\Gamma_\alpha(\epsilon) = 0$, corresponding to the insulating phase. One must check, however, whether this solution is stable with respect to an infinitesimal damping η.

To perform the standard linear stability analysis of Eqs. (18), (19) we linearize Eq. (19) as

$$A_\alpha(\epsilon) = \delta(\epsilon - \xi_\alpha) + \frac{1}{\pi} \frac{\Gamma_\alpha(\epsilon)}{(\epsilon - \xi_\alpha)^2} + O(\Gamma^2) \,, \qquad (24)$$

in complete analogy with Abou-Chacra, Anderson, and Thouless (1973), substitute this into Eq. (18), and obtain a linear integral equation:

$$\Gamma_\alpha(\epsilon) = \eta + \sum_{\beta,\gamma,\delta} |V_{\alpha\beta\gamma\delta}|^2 \frac{2\Gamma_\gamma(\epsilon + \xi_\beta - \xi_\delta) + \Gamma_\beta(\xi_\gamma + \xi_\delta - \epsilon)}{(\epsilon + \xi_\beta - \xi_\gamma - \xi_\delta)^2} \times$$
$$\times \left[n_\beta(1 - n_\gamma)(1 - n_\delta) + (1 - n_\beta)n_\gamma n_\delta \right]. \qquad (25)$$

The solution of this equation may be sought in the form of a perturbations series

$$\Gamma_\alpha(\epsilon) = \sum_{n=0}^{\infty} \Gamma_\alpha^{(n)}(\epsilon) \,, \qquad (26)$$

where $\Gamma^{(n)}$ is of the order $|V|^{2n} \sim (\lambda\delta_\zeta)^{2n}$ and is obtained after n iterations of Eq. (25) starting from $\Gamma^{(0)} = \eta$. Each term in this expansion can be calculated explicitly and its statistics can be determined (details of this rather cumbersome calculation are given in Sec. 6 of BAA paper). The resulting probability distribution function is controlled by a single parameter γ_n, which determines the typical scale of the random quantity $\Gamma^{(n)}$:

$$P_n(\Gamma^{(n)}) = e^{-\gamma_n/\Gamma^{(n)}} \sqrt{\frac{\gamma_n/\pi}{\left[\Gamma^{(n)}\right]^3}} \,, \quad \gamma_n = C_1 \eta \Lambda^{2n} \,, \quad \Lambda \equiv C_2 \frac{\lambda T}{\delta_\zeta} \ln \frac{1}{\lambda} \,. \qquad (27)$$

Here $C_1 \sim 1$ and $C_2 \sim 1$ are model-depndent numerical constants (Eq. (172) of BAA paper contains their values for the specific model adopted there).

The behavior of this distribution function at $n \to \infty$ is qualitatively different, depending on whether Λ in Eq. (27) is smaller or larger than 1, as illustrated by Fig. 4. In the first case the typical scale $\gamma_n \to 0$ as $n \to \infty$, and the distribution function for the total Γ is always concentrated around $\Gamma \sim \eta$ which tends to zero as $\eta \to 0$. This perfectly matches the insulating behavior described in Sec. 3.3 and shown in Fig. 2b.

Oppositely, for $\Lambda > 1$ the scale $\gamma_n \to \infty$; we emphasize that the limit $n \to \infty$ should be taken *prior* to $\eta \to 0$. This means that the distribution function $P(\Gamma)$ does not shrink to $\delta(\Gamma)$ as $\eta \to 0$; in fact, it cannot be determined from the linearized equation (25), the full self-consistent problem (18), (19) should be solved. The divergent linear solution signals the instability of the insulating state and the onset of the metallic state.
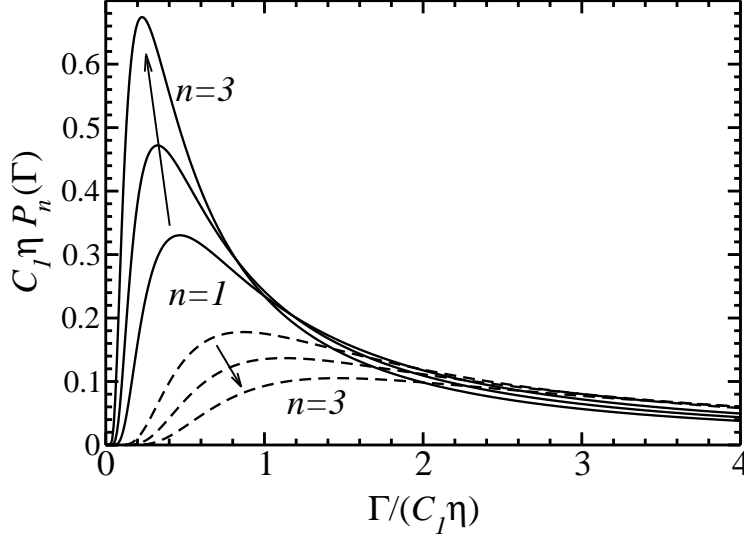
FIG. 4. Plot of the distribution functions $P_n(\Gamma^{(n)})$ as given by Eq. (27) for $n = 1, 2, 3$ for $T/T_c = 0.7$ (solid lines) and $T/T_c = 1.3$ (dashed lines).

These arguments enable us to identify the temperature at which $\Lambda = 1$ with the temperature of the metal-insulator transition:

$$T_c = \frac{\delta_\zeta}{C_2 \lambda \ln(1/\lambda)} \,. \tag{28}$$

4 Metal-insulator transition and many-body mobility edge

In the previous section we considered the decay of a quasiparticle excitation, and found that possibility or impossibility of such decay may be viewed as delocalization or localization of this excitation in the many-electron Fock space. In this section we discuss macroscopic implications of this picture. How can the very notion of localization be applied to many-body states?

Consider a many-body eigenstate $|\Psi_k\rangle$ of the interacting system, with the corresponding eigenenergy E_k. In the coordinate representation, the many-body wave function $\Psi_k \left(\{\vec{r}_j\}_{j=1}^N \right)$ depends on the coordinates of all N particles in the system. Let us create an electron-hole pair on top of $|\Psi_k\rangle$. The resulting state, which is not an eigenstate of the system, can be expanded in terms of other eigenstates:

$$\hat{c}_\alpha^\dagger \hat{c}_\beta \,|\Psi_k\rangle = \sum_{k'} C_{\alpha\beta}^{kk'} \,|\Psi_{k'}\rangle \,; \quad \sum_{k'} \left| C_{\alpha\beta}^{kk'} \right|^2 = 1. \tag{29}$$

It is possible that the number of terms contributing to the sum is effectively finite, *i.e.*

$$\lim_{\mathcal{V}\to\infty}\left[\sum_{k'}\left|C_{\alpha\beta}^{kk'}\right|^4\right]^{-1} < \infty. \tag{30}$$

This corresponds to *insulating* or *localized many-body* state; excitation can not propagate over all states allowed by the energy conservation.

The opposite case, when expansion (29) contains an *infinite* number of eigen-states

$$\lim_{\mathcal{V}\to\infty}\left[\sum_{k'}\left|C_{\alpha\beta}^{kk'}\right|^4\right]^{-1} = \infty, \tag{31}$$

corresponds to *metallic* or *extended many-body* state.

Developed metallic state is formed when expansion (29) involves all the eigen-states with close enough energies:

$$|C_{\alpha\beta}^{kk'}|^2 \propto \text{``}\delta(E_k + \omega_{\alpha\beta} - E_{k'})\text{''}, \tag{32}$$

where δ-function should be understood in the thermodynamic sense: its width, although sufficiently large to include many states, vanishes in the limit $\Omega \to \infty$. Only in this regime, which may also be called *ergodic many-body* state, the electron-electron interaction can bring the system from the initial Hartree-Fock state to the equilibrium corresponding to spanning all the states permitted by the energy conservation. In this case, the averaging over the exact many-body eigenfunction is equivalent to averaging over the microcanonical distribution, and temperature T can be defined as a usual Lagrange multiplier. It is related to E_k by the thermodynamic relation:

$$E_k - E_0 = \int_0^T C_V(T_1)\,dT_1, \tag{33}$$

where E_0 is the ground state energy, and $C_V(T) \propto \Omega$ is the specific heat.

The temperature of the metal-insulator transition, found above, in fact, de-termines the extensive *many-body mobility edge* $\mathcal{E}_c \propto \Omega$. In other words, (i) states with energies $E_k - E_0 > \mathcal{E}_c$ are extended, inelastic relaxation is possible, and the conductivity $\sigma_k = \sigma(E_k)$ in this state is finite; (ii) states with energies $E_k - E_0 < \mathcal{E}_c$ are localized and the conductivity $\sigma(E_k) = 0$.

Let us now assume that the equilibrium occupation is given by the Gibbs distribution. One could think that it would still imply the Arrhenius law (3) for the conductivity. However, this is not the case for the many-body mobility threshold. In fact, in the limit $\Omega \to \infty$

$$\sigma(T) = 0; \quad T < T_c, \tag{34}$$

where the critical temperature is determined by Eq. (33):

$$\int_0^{T_c} dT_1\,C_V(T_1) = \mathcal{E}_c. \tag{35}$$

Therefore, the temperature dependence of the dissipative coefficient in the system shows the singularity typical for a phase transition.

To prove the relations (34), (35) we use the Gibbs distribution and find

$$\sigma(T) = \sum_k P_k \sigma(E_k) = \frac{\int_0^\infty dE\, e^{S(E)-E/T}\sigma(E)}{\int_0^\infty dE\, e^{S(E)-E/T}},$$

where the entropy $S(E)$ is proportional to volume, and E is counted from the ground state. The integral is calculated in the saddle point or in the steepest decent approximations, exact for $\Omega \to \infty$. The saddle point $E(T)$ is given by

$$\left.\frac{dS}{dE}\right|_{E=E(T)} = \frac{1}{T}\,.$$

Taking into account $\sigma(E) = 0$ for $E < \mathcal{E}_c$ we find

$$\sigma(T) = \sigma\left[E(T)\right], \quad E(T) > \mathcal{E}_c;$$
$$\sigma(T) \propto \exp\left(-\frac{\mathcal{E}_c - E(T)}{T}\right); \quad E(T) < \mathcal{E}_c$$

As both energies entering the exponential are extensive, $E(T), \mathcal{E}_c \propto \Omega$, we obtain (34), (35).

To be able to establish the thermal equilibrium in such insulating state the system should be coupled to an external bath (*i.e.*, phonons). The presence of the finite electron-phonon interaction (as phonons are usually delocalized), smears out the transition, and $\sigma(T)$ becomes finite for any temperature. Nevertheless, if electron-phonon interaction is weak, the phenomenon of the many-body metal-insulator transition manifests itself as a sharp crossover from phonon induced hopping at $T < T_c$ to the conductivity independent of the electron-phonon coupling at $T > T_c$.

5 Conclusions and perspectives

We have considered the inelastic relaxation and transport at low temperatures in disordered conductors where all single-particle states are localized, and no coupling to phonons or any other thermal bath is present. The main question is whether electron-electron interaction alone is sufficient to cause transitions between the localized states producing thereby a finite conductivity. The answer to this question turns out to be determined by the Anderson localization-delocalization physics in the many-particle Fock space, and is summarized on Fig. 5.

It should be emphasized that the many-body localization, which we discuss in this paper, is qualitatively different from conventional finite temperature Metal to Insulator transitions, such as formation of a band insulator due to the structural phase transition or Mott-Hubbard transition. In these two cases, at a certain temperature T^* a gap appears in the spectrum of charge excitation (Mott
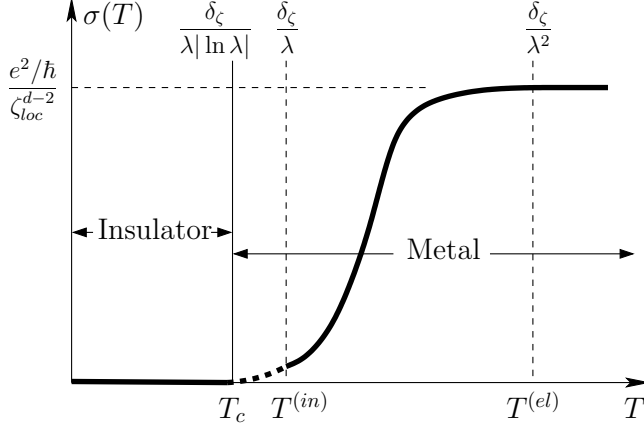
FIG. 5. Schematic temperature dependence of the dc conductivity $\sigma(T)$. Below the point of the many-body metal-insulator transition, $T < T_c$, no inelastic relaxation occurs and $\sigma(T) = 0$. Temperature interval $T \gg T^{(in)} > T_c$ corresponds to the developed metallic phase, where Eq. (21) is valid. At $T \gg T^{(el)}$ the high-temperature metallic perturbation theory (Altshuler and Aronov, 1985) is valid, and the conductivity is given by the Drude formula.

insulator) or all excitations (band insulator). In both cases the conductivity remains finite although exponentially small as long as $T > 0$. This is not the case for many-body localization, which implies exactly zero conductivity in the low-temperature phase.

Is the many-body localization a true thermodynamical phase transition with corresponding singularities in all equilibrium properties? This question definitely requires additional studies, however, some speculations can be put forward. The physics described in the present paper is associated with the change of the characteristics of the many-body wavefunctions. It is well known that for non-interacting systems localization-delocalization transition does not affect the average density of states, *i.e.*, it does not manifest itself in any macroscopic thermodynamic properties. Application of the same logic to the exact many-body eigenvalues would indicate that the many-body localization transition is not followed by any singularities in the static specific heat, etc. On the other hand, at this point we can not rule out the possibility that this conclusion is an artifact of treating the real parts of the electron self-energies with an insufficient accuracy. Most likely scenario, to our opinion, is that the insulating phase behaves like a glass (spin or structural) and demonstrates all the glassy properties (Fisher and Hertz, 1991; Bouchaud *et al.*, 1998), like absence of ergodicity (even when some coupling with phonons is included), effects of aging, etc. Discussion of the equilibrium susceptibilities in the latter case becomes quite meaningless.

The quantitative theory built in BAA paper assumes that the interaction is weak. On the other hand, qualitative consideration of the localization of many-

body excitations does not rely on this assumption. The important ingredients are (i) localization of single-particle excitations, and (ii) Fermi statistics. Consider, as an example, Wigner crystal (Wigner, 1934). It is well known that strong enough interaction leads to a spontaneous breaking of the translational symmetry in d-dimensional clean systems at $d \geq 2$. In a clean system Wigner crystallization is either a first-order phase transition ($d = 3$), or a Kosterlitz-Thouless transition ($d = 2$). Even weak disorder destroys both translational and orientational order (Larkin, 1970) and pins the crystal. The symmetry of this state is thus not different from the symmetry of a liquid, and the thermodynamic phase transition is commonly believed to be reduced to a crossover.

We argue that the many-body localization provides the correct scenario for the finite-temperature "melting" transition between the insulating phase, which may be called "solid", and the metallic phase, which may be called "liquid". Indeed, the conductivity of the pinned Wigner crystal is provided by the motion of defects. At low temperatures and in the absence of the external bath, all defects are localized by the one-particle Anderson mechanism. Phonon modes of the Wigner crystal are localized as well, so the system should behave as a many-body insulator. As the temperature is increased, the many-body metal-insulator transition occurs. It is not clear at present, whether it occurs before or after the crystalline order is destroyed at distances smaller than Larkin's scale. Construction of effective theory of such a transition is a problem which deserves further investigation.

REFERENCES

Abou-Chacra, R., Anderson, P. W., and Thouless, D. J. (1973). *Journal of Physics C*, **6**: 1734.

Abrahams, E., Anderson, P. W., Licciardello, D. C., and Ramakrishnan, T. V. (1979). *Physical Review Letters*, **42**: 673.

Altshuler, B. L., and Aronov, A. G. (1985). In A. L. Efros and M. Pollak (eds.), *Electron-Electron Interactions in Disordered Systems*. Amsterdam: North-Holland, Amsterdam.

Altshuler, B. L., Gefen, Y., Kamenev, A., and Levitov, L. S. (1997). *Physical Review Letters*, **78**: 2803.

Anderson, P. W (1958). *Physical Review*, **109**: 1492.

Basko, D. M., Aleiner, I. L., and Altshuler, B. L. (2006). *Annals of Physics*, forthcoming.

Berezinskii, V. L. (1973). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **65**: 1251 [*Soviet Physics: JETP*, **38**: 620].

Bouchaud, J.-P., Cugliandolo, L. F., Kurchan, J., and Mezard, M. (1998). In A. P. Young (ed.), *Spin Glasses and Random Fields*. Singapore: World Scientific.

Dorokhov, O. N. (1983). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **85**: 1040 [*Soviet Physics: JETP*, **58**: 606].

Efetov, K. B. (1987). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **92**: 638; *ibid.* **93**: 1125 [*Soviet Physics: JETP*, **65**: 360; *ibid.* **66**: 634].

Efetov, K. B., and Larkin, A. I. (1983). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **85**: 764 [*Soviet Physics: JETP*, **58**: 444].

Eliashberg, G. M. (1962). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* **42**: 1658 [*Soviet Physics: JETP* **15**: 1151].

Fischer, K. H. and Hertz, J. A. (1991). *Spin glasses*. Cambridge: Cambridge University Press.

Fleishman, L., and Anderson, P. W. (1980). *Physical Review B*, **21**: 2366.

Fritzsche, H. (1955). *Physical Review*, **99**: 406.

Gertsenshtein, M. E., and Vasil'ev, V. B. (1959). *Teoriya Veroyatnostei i ee Primeneniya*, **4**: 424; *ibid.*, **5**: 3(E) [*Theory of Probability and its Applications*, **4**: 391; *ibid.*, **5**, 340(E)].

Gogolin, A. A., Melnikov, V. I., and Rashba, É. I. (1975). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **69**: 327 [*Soviet Physics: JETP*, **42**: 168].

Gornyi, I. V., Mirlin, A. D., and Polyakov, D. G. (2004). cond-mat/0407305, version 1.

Keldysh, L. V. (1964). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **47**: 1945 [*Soviet Physics: JETP*, **20**: 1018].

Kozub, V. I., Baranovskii, S. D., and Shlimak, I. (2000). *Solid State Communications*, **113**: 587.

Landau, L. D, (1958). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **35**: 97 [*Soviet Physics: JETP*, **8**: 70].

Larkin, A. I. (1970). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **58**: 1466 [*Soviet Physics: JETP*, **31**: 784].

Mott, N. F. (1968a). *Journal of Non-Crystalline Solids*, **1**: 1.

Mott, N. F. (1968b). *Phylosophical Magazine*, **17**: 1259.

Nattermann, T., Giamarchi, T., and Le Doussal, P. (2003). *Physical Review Letters*, **91**: 056603.

Shahbazyan, T. V., and Raikh, M. E. (1996). *Physical Review B*, **53**: 7299.

Shklovskii, B. I., and Efros, A. L. (1984). *Electronic Properties of Doped Semiconductors*. Berlin: Springer-Verlag.

Thouless, D. J. (1977). *Physical Review Letters*, **39**: 1167.

Wigner, E. P. (1934). *Physical Review*, **46**: 1002.